PMM U.S.S.R., Vol.54, No.6, pp.765-768, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

THE BRACHISTOCHRONE MOTION OF A MECHANICAL SYSTEM WITH NON-HOLONOMIC, NON-LINEAR AND NON-STATIONARY CONSTRAINTS*

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Further to previous studies /1, 2/ of the brachistochrone motion of non-holonomic mechanical systems with linear homogeneous constraints, consideration is given here to non-holonomic, non-linear and non-stationary mechanical systems. The problem is to formulate the differential equations of the brachistochrone motion of non-holonomic, non-linear and non-stationary mechanical systems and to determine the additional forces which must be introduced in order to implement motion of this type.

1. Let us analyse the constraints imposed on a mechanical system which has to be moved from position A to position B in minimum time. The mechanical system is moving in a known force field. The additional forces satisfy the condition

$$R_\alpha q^{\cdot\alpha} = 0 \tag{1.1}$$

Thus, they do not affect the law governing the variation of the total mechanical energy of the system. Here and below repeated indices will indicate summation. The indices have the following ranges: $i, j, k, s = 1, \ldots, n$; $\alpha, \beta, = 1, \ldots, m$; $\nu, \rho = m+1, \ldots, m+l$.

The constraints in the second group are non-holonomic, non-linear and non-stationary:

$$q^{\cdot\nu} = \psi^\nu (q^i, q^{\cdot\alpha}, t). \tag{1.2}$$

In addition to (1.1) and (1.2), we introduce two additional relations (see my doctorate dissertation** and /3/):

$$p_s - \frac{\partial T}{\partial q^{\cdot s}} = 0 \tag{1.3}$$

$$p_\alpha^{\cdot} - \frac{\partial T}{\partial q^\alpha} + \frac{\partial \Pi}{\partial q^\alpha} - Q_\alpha + \frac{\partial \psi^\rho}{\partial q^{\cdot\alpha}} \left(p_\rho - \frac{\partial T}{\partial q^\rho} + \frac{\partial \Pi}{\partial q^\rho} - Q_\rho\right) - R_\alpha = 0 \tag{1.4}$$

Here T is the kinetic energy of the system:

$$T = {}^1/_2 a_{ij}(q^k) q^{\cdot i} q^{\cdot j}$$

$\Pi(q^k)$ is the potential energy, Q_i are non-conservative forces which depend in the most general case on the generalized coordinates, generalized velocities and time, and R_α are additional and as yet undertermined forces.

The starting position A of the system is specified in terms of the time t_0 and generalized coordinates $q_{(0)}^i$, and the terminal position B by the time t_1 (as yet undetermined) and generalized coordinates $q_{(1)}^i$. At the starting position $q_{(0)}^i$ the following quantity is given:

$$T + \varepsilon^\nu p_\nu \tag{1.5}$$

$$\varepsilon^\nu = \frac{\partial \psi^\nu}{\partial q^{\cdot\alpha}} q^{\cdot\alpha} - \psi^\nu, \quad p_\nu = \frac{\partial T}{\partial q^{\cdot\nu}}$$

In the case of homogeneous constraints $\varepsilon^\nu = 0$, and therefore (1.5) is simply the kinetic energy of the system. In the general case this quantity has the form $T + \varepsilon^\nu p_\nu = T_2^* - T_0^*$, where $T_2^* = T_2 |_{q^{\cdot\nu} = \psi^\nu}$ is the quadratic part of the kinetic energy and T_0^* is the part of the kinetic energy independent of the generalized velocities.

The time required for the system to move from A to B is given by the integral

$$J = \int_{t_0}^{t_1} dt \tag{1.6}$$

*Prikl. Matem. Mekhan., 54, 6, 931-935, 1990

** Zeković D., Neki problemi dinamike neholonomnih sistema sa primenom na tehničke objekte Doctorate Dissertation, Beograd, 1984.

This integral has to be minimized. When it is a minimum, the system is performing brachistochrone motion. The problem is to determine the conditions for the functional (1.6) to have an extreme point, subject to the constraints (1.1)-(1.4). This variational problem may be phrased equivalently as follows: find the conditions for the following new functional to have an extreme point /4/:

$$\bar{J} = \int_{t_0}^{t_1} \bar{F}\, dt \tag{1.7}$$

where the integrand is

$$\bar{F} = 1 + \lambda R_\alpha q^{\cdot\alpha} + \mu^s\left(p_s - \frac{\partial T}{\partial q^{\cdot s}}\right) + \nu^\alpha\left[p_\alpha^{\cdot} - \frac{\partial T}{\partial q^\alpha} + \frac{\partial \Pi}{\partial q^\alpha} - Q_\alpha + \frac{\partial \psi^\rho}{\partial q^{\cdot\alpha}}\left(p_\rho^{\cdot} - \frac{\partial T}{\partial q^\rho} + \frac{\partial \Pi}{\partial q^\rho} - Q_\rho\right) - R_\alpha\right] + \theta_\rho\left(q^{\cdot\rho} - \psi^\rho\right) \tag{1.8}$$

λ, μ^s, ν^α, θ_ρ being Lagrange multipliers of the constraints.

According to the rules of variational calculus, the conditions for the functional (1.7) to have an extreme points reduce to the following equations /5/:

$$\frac{d}{dt}\frac{\partial \bar{F}}{\partial R_\alpha^{\cdot}} - \frac{\partial \bar{F}}{\partial R_\alpha} = 0 \rightarrow \nu^\alpha - \lambda q^{\cdot\alpha} = 0 \tag{1.9}$$

$$\frac{d}{dt}\frac{\partial \bar{F}}{\partial p_\alpha^{\cdot}} - \frac{\partial \bar{F}}{\partial p_\alpha} = 0 \rightarrow \nu^{\cdot\alpha} - \mu^\alpha = 0 \tag{1.10}$$

$$\frac{d}{dt}\frac{\partial \bar{F}}{\partial p_\nu^{\cdot}} - \frac{\partial \bar{F}}{\partial p_\nu} = 0 \rightarrow \frac{d}{dt}\left(\frac{\partial \psi^\nu}{\partial q^{\cdot\alpha}}\nu^\alpha\right) - \mu^\nu = 0 \tag{1.11}$$

$$\frac{d}{dt}\frac{\partial \bar{F}}{\partial q^{\cdot\alpha}} - \frac{\partial \bar{F}}{\partial q^\alpha} = 0, \quad \frac{d}{dt}\frac{\partial \bar{F}}{\partial q^{\cdot\nu}} - \frac{\partial \bar{F}}{\partial q^\nu} = 0 \tag{1.12}$$

$$\left(\frac{\partial \bar{F}}{\partial p_i^{\cdot}}\delta p_i + \frac{\partial \bar{F}}{\partial R_\alpha^{\cdot}}\delta R_\alpha + \frac{\partial \bar{F}}{\partial q^{\cdot i}}\partial q^i\right)\Big|_{t_0}^{t_1} = 0 \rightarrow \lambda\left(q^{\cdot\alpha}\delta p_\alpha + \frac{\partial \psi^\rho}{\partial q^{\cdot\alpha}}q^{\cdot\alpha}\delta p_\rho\right)\Big|_{t_0}^{t_1} = 0 \tag{1.13}$$

$$\left[\bar{F} - \left(\frac{\partial \bar{F}}{\partial p_i^{\cdot}}p_i^{\cdot} + \frac{\partial \bar{F}}{\partial R_\alpha^{\cdot}}R_\alpha^{\cdot} + \frac{\partial \bar{F}}{\partial q^{\cdot i}}q^{\cdot i}\right)\right]\delta t\Big|_{t_0}^{t_1} = 0 \tag{1.14}$$

Eqs.(1.9)-(1.12) are the Euler equations of the variational problem with integrand (1.8). Eq.(1.13) is the natural boundary condition, while Eq.(1.14) is the transversality condition at the right end.

Eqs.(1.12) may be written explicitly as follows:

$$\frac{d}{dt}\frac{\partial F^*}{\partial q^{\cdot\alpha}} - \frac{\partial F^*}{\partial q^\alpha} - \frac{\partial \psi^\rho}{\partial q^{\cdot\alpha}}\frac{\partial F}{\partial q^\rho} + \left(\frac{\partial F}{\partial q^\rho} - \bar{\theta}_\rho\right)\gamma_\alpha{}^\rho - \lambda\left(\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\nu}} - \frac{\partial T}{\partial q^\nu} + \frac{\partial \Pi}{\partial q^\nu}\right)(\gamma_\alpha{}^\nu - \delta_\alpha{}^\nu) + N_\alpha{}^* = 0 \tag{1.15}$$

$$\frac{d}{dt}\frac{\partial F}{\partial q^{\cdot\nu}} - \frac{\partial F}{\partial q^\nu} - \bar{\theta}_\nu^{\cdot} - \theta_\rho\frac{\partial \psi^\rho}{\partial q^\nu} + N_\nu = 0$$

The new variables F, F^*, $\bar{\theta}_\nu$, $\gamma_\alpha{}^\rho$, $\delta_\alpha{}^\rho$, $N_\alpha{}^*$ and N_ν in Eqs.(1.15) are:

$$F = \lambda^{\cdot}T + \lambda\bar{Q}_i q^{\cdot i} \quad \left(\bar{Q}_i = Q_i + Q_i{}^K = Q_i - \frac{\partial \Pi}{\partial q^i}\right) \tag{1.16}$$

$$F^* = F\big|_{q^{\cdot\nu} = \psi^\nu} = \lambda^{\cdot}T^* + \lambda\bar{Q}_\beta{}^* q^{\cdot\beta}$$

$$\left[\bar{Q}_\beta{}^* = Q_\beta{}^* + Q_\beta^{*K} = Q_\beta{}^* - \left(\frac{\partial \Pi}{\partial q^\beta} + \frac{\partial \psi^\rho}{\partial q^{\cdot\beta}}\frac{\partial \Pi}{\partial q^\rho}\right)\right]$$

$$\bar{\theta}_\nu = \theta_\nu - \lambda\left(\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\nu}} - \frac{\partial T}{\partial q^\nu} + \frac{\partial \Pi}{\partial q^\nu} - Q_\nu\right)$$

$$\gamma_\alpha{}^\rho = \left[\frac{\partial \psi^\rho}{\partial q^\alpha} + \frac{\partial \psi^\nu}{\partial q^{\cdot\alpha}}\frac{\partial \psi^\rho}{\partial q^\nu} - \frac{d}{dt}\left(\frac{\partial \psi^\rho}{\partial q^{\cdot\alpha}}\right)\right]$$

$$\delta_\alpha{}^\rho = \left[\frac{\partial^2 \psi^\rho}{\partial q^{\cdot\beta}\partial q^\alpha}q^{\cdot\beta} + \frac{\partial^2 \psi^\rho}{\partial q^{\cdot\beta}\partial q^\nu}\frac{\partial \psi^\nu}{\partial q^{\cdot\alpha}}q^{\cdot\beta} - \frac{d}{dt}\left(\frac{\partial \psi^\rho}{\partial q^{\cdot\alpha}}\right)\right]$$

$$N_\alpha{}^* = -\varepsilon^\rho\left[\lambda\frac{\partial}{\partial q^\alpha}\left(2\frac{\partial \Pi}{\partial q^\rho} - \frac{\partial T}{\partial q^\rho} + Q_\rho\right) - \lambda\frac{\partial Q_\rho}{\partial q^{\cdot\alpha}}\right] + \frac{d}{dt}\left\{\left[\lambda\left(\frac{d}{dt}\frac{\partial h_\alpha}{\partial q^{\cdot\rho}} - \frac{\partial h_\alpha}{\partial q^\rho}\right)\varepsilon^\rho + \frac{d}{dt}(\lambda a_{\rho\alpha}\varepsilon^\rho)\right]\right\} +$$

$$\Big[\lambda q^{\cdot\beta}\frac{\partial^2\psi^\rho}{\partial q^{\cdot\alpha}\partial q^{\cdot\beta}}\Big(\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\rho}}-\frac{\partial T}{\partial q^\rho}+\frac{\partial\Pi}{\partial q^\rho}-Q_\rho\Big)+\lambda q^{\cdot\beta}Q_\rho\frac{\partial^2\psi^\rho}{\partial q^{\cdot\alpha}\partial q^{\cdot\beta}}\Big]-$$
$$\frac{\partial\psi^\nu}{\partial q^{\cdot\alpha}}\Big\{\varepsilon^\rho\Big[\lambda\frac{\partial}{\partial q^\nu}\Big(2\frac{\partial\Pi}{\partial q^\rho}-\frac{\partial T}{\partial q^\rho}+Q_\rho\Big)-\lambda\frac{\partial Q_\rho}{\partial q^{\cdot\nu}}\Big]\Big\}-$$
$$\frac{\partial\psi^\nu}{\partial q^{\cdot\alpha}}\frac{d}{dt}\Big\{\frac{d}{dt}(\lambda a_{\nu\rho}\varepsilon^\rho)-\lambda\varepsilon^\rho\Big(\frac{d}{dt}\frac{\partial h_\rho}{\partial q^{\cdot\nu}}-\frac{\partial h_\rho}{\partial q^\nu}\Big)\Big\}$$
$$N_\nu=-\varepsilon^\rho\Big[\lambda\frac{\partial}{\partial q^\nu}\Big(2\frac{\partial\Pi}{\partial q^\rho}-\frac{\partial T}{\partial q^\rho}+Q_\rho\Big)-\lambda\frac{\partial Q_\rho}{\partial q^{\cdot\nu}}\Big]-$$
$$\frac{d}{dt}\Big\{\frac{d}{dt}(\lambda a_{\nu\rho}\varepsilon^\rho)-\lambda\varepsilon^\rho\Big(\frac{d}{dt}\frac{\partial h_\rho}{\partial q^{\cdot\nu}}-\frac{\partial h_\rho}{\partial q^\nu}\Big)\Big\},\quad h_s=\frac{\partial f}{\partial q^{\cdot s}}$$

After some reduction, Eq.(1.14) becomes

$$\Big\{1+\frac{d}{dt}\Big[\lambda q^{\cdot\alpha}\Big(p_\alpha+\frac{\partial\psi^\rho}{\partial q^{\cdot\alpha}}p_\rho\Big)+2\lambda\Big[\frac{\partial\Pi}{\partial q^\alpha}q^{\cdot\alpha}+\frac{\partial\psi^\rho}{\partial q^{\cdot\alpha}}\frac{\partial\Pi}{\partial q^\rho}q^{\cdot\alpha}-Q_\alpha q^{\cdot\alpha}-\tag{1.17}$$
$$\frac{\partial\psi^\rho}{\partial q^{\cdot\alpha}}Q_\rho q^{\cdot\alpha}\Big]+\lambda\Big(\frac{\partial Q_\alpha}{\partial q^{\cdot j}}q^{\cdot\alpha}q^{\cdot j}+\frac{\partial\psi^\rho}{\partial q^{\cdot\alpha}}\frac{\partial Q_\rho}{\partial q^{\cdot j}}q^{\cdot\alpha}q^{\cdot j}\Big)-$$
$$\lambda q^{\cdot\alpha}\frac{\partial^2\psi^\rho}{\partial q^{\cdot\alpha}\partial q^{\cdot\beta}}\Big(p_\rho-\frac{\partial T}{\partial q^\rho}+\frac{\partial\Pi}{\partial q^\rho}-Q_\rho\Big)q^{\cdot\beta}+\theta_\rho\varepsilon^\rho\Big\}\delta t\Big|_{t=t_1}=0$$

In sum, we obtain the following version of the problem: solve Eqs.(1.2) and (1.15) and the following equation, which follows from (1.1) and (1.4):

$$\frac{d}{dt}(T+\Pi)-Q_iq^{\cdot i}+\Big(\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\nu}}-\frac{\partial T}{\partial q^\nu}+\frac{\partial\Pi}{\partial q^\nu}-Q_\nu\Big)\Big(\psi^\nu-\frac{\partial\psi^\nu}{\partial q^{\cdot\alpha}}q^{\cdot\alpha}\Big)=0\tag{1.18}$$

This system of equations contains one second-order Eq.(1.18), l first-order Eqs.(1.2), m third-order equations and l second-order Eqs.(1.15) in the $m+2l+1=n+l+1$ unknowns q^i, θ_ρ and λ. The change of variables $q^{\cdot\alpha}=y^\alpha$, $y^{\cdot\alpha}=x^\alpha$, $\lambda^\cdot=\Lambda$ reduces it to a system of first-order differential equations containing the same number of constants of integration. The constants of integration are determined by $2n$ boundary conditions $q_{(0)}{}^i$, $q_{(1)}{}^i$, and the natural boundary condition at the left end (1.13), which may be reduced to the equation

$$\lambda\Big[q^{\cdot\alpha}\delta p_\alpha+q^{\cdot\nu}\delta p_\nu+\Big(\frac{\partial\psi^\nu}{\partial q^{\cdot\alpha}}q^{\cdot\alpha}-\psi^\nu\Big)\delta p_\nu\Big]\Big|_{t=t_0}=0$$

Now, noting tht the quantity (1.5) is given, we obtain

$$\lambda\frac{\partial^2\psi^\nu}{\partial q^{\cdot\alpha}\partial q^{\cdot\beta}}q^{\cdot\alpha}p_\nu\Big|_{t=t_0}=0$$

Similarly,

$$\lambda\Big[q^{\cdot\alpha}\delta p_\alpha+q^{\cdot\nu}\delta p_\nu+\Big(\frac{\partial\psi^\nu}{\partial q^{\cdot\alpha}}{}'q^{\cdot\alpha}-\psi^\nu\Big)\delta p_\nu\Big]\Big|_{t=t_1}=0$$

Since (1.5) is not fixed at the right end, it follows that $\lambda_{(t_1)}=0$. The transversality condition (1.17) at the right end gives one more equation for the arbitrary constants.

Thus, we have $2n+m+2=3m+2l+2$ equations for the same number of constants of integration. Since the time t_1 must also be determined, we can take (1.5) as an additional condition.

Once the functions $q^i=q^i(t)$ have been determined, the additional forces are found from the equations

$$R_\alpha=\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\alpha}}-\frac{\partial T}{\partial q^\alpha}+\frac{\partial\Pi\cdot}{\partial q^\alpha}-Q_\alpha+\frac{\partial\psi^\rho}{\partial q^{\cdot\alpha}}\Big(\frac{d}{dt}\frac{\partial T}{\partial q^{\cdot\rho}}-\frac{\partial T}{\partial q^\rho}+\frac{\partial\Pi}{\partial q^\rho}-Q_\rho\Big)$$

2. As an example, let us consider two heavy material points M_1 and M_2 of unit mass, connected by a rod of fixed length $2l$ and moving in a vertical plane in such a way that the velocity of the midpoint **v** of the rod is perpendicular to the segment M_1M_2. This non-holonomic constraint can be implemented by means of a knife-edge, as in the case of the Chaplygin sleight /6/ (see Fig.1).

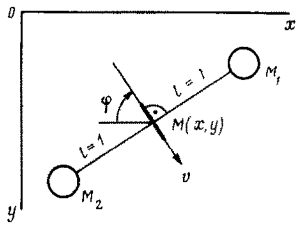

Fig.1

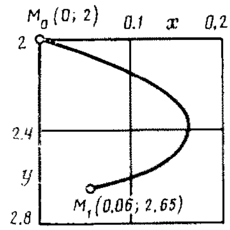

Fig.2

Let x, y be the coordinates of the point M, and φ the angle of rotation of the knife-edge (Fig.1). In the case under consideration the total energy is conserved (we may assume that its constant value is zero). Therefore the functional (1.7) becomes

$$\int_{t_0}^{t_1}[1+\Lambda(T+\Pi)+\theta(y^\cdot-x^\cdot\operatorname{tg}\varphi)]\,dt=\int_{t_0}^{t_1}F_1\,dt \tag{2.1}$$

and the equations of the constraints may be taken as $y^\cdot - x^\cdot\operatorname{tg}\varphi=0,\ T+\Pi=0$. The extremality condition for the functional (2.1) yields the following equations:

$$2\Lambda x^\cdot-\theta\operatorname{tg}\varphi=c_1\quad(2\Lambda y^\cdot+\theta)^\cdot+2\Lambda g=0\quad(2\Lambda\varphi^\cdot)^\cdot+\theta x^\cdot\cos^{-2}\varphi=0 \tag{2.2}$$

Eqs.(2.2), together with the two equations of the constraints, form a closed system of equations for the coordinates x, y, φ and multipliers Λ and θ. These equations also involve an as yet undertermined parameter c_1.

The problem was solved numerically with the following data: starting position $x_0=0$, $y_0=2$, $\varphi_0=0.785$, terminal position $x_1=0.06$, $y_1=2.65$, $\varphi_1=2.628$ and starting velocities $x_0^\cdot=2$, $y_0^\cdot=2$, $\varphi_0^\cdot=5.41$. By varying the constant c_1, we found the required brachistochrone curve passing through the point $(x_1, y_1\ \varphi_1)$; this turned out to correspond to the value $c_1=0.001$. The minimum time for the system to move from (x_0, y_0, φ_0) to (x_1, y_1, φ_1) was 0.299 sec. The trajectory of the point M is shown in Fig.2.

I wish to express my thanks to V. Čović for his interest and comments.

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Translated by D.L.